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RADIATION TRANSFER THROUGH SPECULAR PASSAGES—A SIMPLE APPROXIMATION*

ARI RABL
Solar Energy Group, Argonne National Laboratory, 9700 South Cass Avenue, Argonne, Il 60439, U.S.A.

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Abstract—A new technique is developed for approximating exchange factors for specular radiation passages. It is shown that for a large class of configurations, even with curved reflector surfaces, the average number of reflections $\langle n\rangle$ can be calculated by a simple analytic formula and *without any ray tracing*. The fraction of radiation transmitted through the passage, τ (one of the exchange factors), can then be approximated by $\tau \approx \rho^{\langle n\rangle}$ if the specular reflectivity ρ of the passage wall is high. A rigorous lower bound is derived which agrees with the exact result within a few percent for any $\varepsilon = 1-\rho$, provided $\varepsilon\langle n\rangle$ is not too large. Several examples are discussed, including cylindrical passages, V-troughs and compound parabolic concentrators. The method is particularly useful for calculating transmission and absorption of radiation in solar concentrators.

NOMENCLATURE

A, area, with appropriate subscripts, e.g. A_R = reflector area;
C, concentration, $= A_A/A_B$;
$E(\rho)$, exchange factor, with appropriate subscripts, e.g. $E_{B-A}(\rho)$ = exchange factor for radiation from B to A;
F, shape factor, with appropriate subscripts, e.g. F_{B-A} = shape factor for radiation from B to A;
$\langle n\rangle$, average number of reflections, with appropriate subscripts, e.g. $\langle n\rangle_{B-A}$ = average number of reflections for radiation from B to A;
$P(n)$, probability of n reflections, with appropriate subscripts, e.g. $P_{B-A}(n)$ = probability of n reflections for radiation from B to A;
Q, radiation heat transfer (power), with appropriate subscripts, e.g. $Q_{B\to A}$ = transfer from B to A, and $Q_{B\rightleftarrows A}$ = net transfer between B and A;
T, absolute temperature, with appropriate subscripts, e.g. T_R = temperature of reflector;
ε, emissivity (absorptivity) of reflector;
ρ, $= 1-\varepsilon$, reflectivity of reflector;
σ, Stefan–Boltzmann constant;
τ, transmission factor, $= E_{A-B}(\rho)$.

The subscripts A, B and R refer to entrance aperture, exit aperture and reflector wall, respectively. The quantities l, h, r, $\eta = l/r$, ϕ, θ designate characteristic dimensions or angles of radiation passages.

1. INTRODUCTION

RADIATION heat exchange in enclosures with specular surfaces can be extremely difficult to calculate, especially if the specular surfaces are curved. Only in the last two decades has this problem been studied systematically [1–3]. The introduction of the exchange factor was a very significant advance since it provides an elegant general framework for the analysis of radiation transfer between surfaces which are partially specular and partially diffuse. The evaluation of the exchange factors themselves, however, has remained a tedious task, generally involving the summation of a large or infinite number of reflections. For this reason many calculations rely on computers to carry out the ray tracing. Such numerical methods can consume a great deal of computing time in system optimization studies. On the other hand, a simple analytic formula, even if only approximate, may be valuable not only because it saves computer time, but also because it can provide better intuitive understanding and serve as guide in the selection of a good design.

This article presents a new approach which is based on the concept of the average number of reflections $\langle n\rangle$, and provides a simple approximation for the transmission factor τ, i.e. the fraction of radiation which is transmitted through a specular passage. In the notation of [3] and Fig. 1, τ is an exchange factor, $\tau = E_{A-B}(\rho)$. Since exchange factors depend only on the specular component ρ_s of the reflectivity, we shall assume $\rho = \rho_s$ throughout this paper. $\langle n\rangle$ is defined with respect to radiation leaving an emitter (rather than arriving at an absorber) and hence it depends

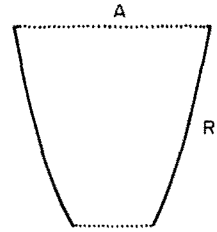

FIG. 1. Radiation passage; entrance aperture A, reflector wall R, exit aperture B.

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only on geometry but not on surface properties. Two formulas are derived which are useful whenever $\langle n \rangle$ is sufficiently small (generally $\langle n \rangle \lesssim 1$). The first one, $\tau \simeq \rho^{\langle n \rangle}$, is appropriate if the reflectivity ρ of the passage wall is high. The second one, just slightly more complicated, is actually a rigorous lower bound which agrees with the exact result to within a few percent when $\varepsilon\langle n \rangle \lesssim 0.1$, for any $\varepsilon = 1-\rho$. For some configurations, good agreement is obtained even for $\varepsilon\langle n \rangle \gtrsim 1$.

The paper is organized as follows: In Section 2, the connection between exchange factors and average number of reflections $\langle n \rangle$ is established by considering the radiation balance of the reflector, and the approximation $\tau \simeq \rho^{\langle n \rangle}$ and the rigorous lower bound are justified. The remaining sections deal with examples where $\langle n \rangle$ can be calculated without ray tracing. In Sections 3 and 4, respectively, the approximations are compared numerically with the exact solution for circular cylindrical passages and for compound parabolic concentrators. Further examples; parallel straight reflectors, cylindrical reflectors, V-troughs, modified compound parabolic concentrators and convolute reflectors for tubes, are discussed in Section 5. We close, in Section 6, with some comments on the general applicability of this technique.

2. RADIATION TRANSFER THROUGH SPECULAR PASSAGES AND AVERAGE NUMBER OF REFLECTIONS

It is convenient to consider a radiation passage as a three-surface enclosure, as shown in Fig. 1, where R is specular, and A and B are diffuse. Since the formalism for calculating heat transfers in terms of exchange factors is well understood [3], we shall concern ourselves only with the evaluation of the latter. The exchange factor is defined as that fraction of the diffuse radiation leaving one surface which reaches another surface either directly or via any intervening specular reflections. For example, $E_{B-A}(\rho_s)$ for the passage in Fig. 1 is the fraction of the diffuse radiation emanating from B which reaches A directly or after specular reflection(s) off R; ρ_s is the specular component of the reflectivity of R. Exchange factors are functions only of geometry and of the relevant specular reflectivities, in particular, they are independent of any diffuse components of reflectivity. Therefore, the exchange factors for the passage in Fig. 1 can be calculated by assuming R to be purely specular, $\rho = \rho_s$, while A and B are perfectly black; this is a substantial simplification of the problem.

Let us analyze the radiation transfers in Fig. 1, using the notation Q for power, A for area, and T for absolute temperature, with appropriate subscripts for each of the three surfaces A, B and R. Of the radiation

$$Q_A = A_A \sigma T_A^4 \tag{1}$$

emitted by A a fraction $E_{A-B}(\rho)$ arrives at B while another fraction, $E_{A-A}(\rho)$, is sent back to A. By energy conservation, the remainder

$$Q_{A\to R} = [1 - E_{A-B}(\rho) - E_{A-A}(\rho)] A_A \sigma T_A^4 \tag{2}$$

must have been absorbed by the reflector R. If the reflector is a diffuse emitter emissivity $\varepsilon = 1-\rho$, then the radiation transfer from R to A is

$$Q_{R\to A} = E_{R-A}(\rho) A_R \varepsilon \sigma T_R^4 \tag{3}$$

$E_{R-A}(\rho)$ being, of course, the fraction of radiation from R which reaches A. By the second law of thermodynamics, the net transfer

$$Q_{A\rightleftarrows R} = Q_{A\to R} - Q_{R\to A} \tag{4}$$

must vanish if the temperatures T_A and T_R are equal. Thus, one can extract the relation (c.f. equation (9–44) of [4])

$$1 - E_{A-B}(\rho) - E_{A-A}(\rho) = \varepsilon \frac{A_R}{A_A} E_{R-A}(\rho). \tag{5}$$

For the transfer between B and R, we assume that

$$E_{B-B}(\rho) = 0, \tag{6}$$

i.e. no radiation emitted by B returns to B; this is certainly true for passages whose width increases monotonically from B to A, as suggested in Fig. 1 or for passages of constant width as for example Fig. 3. We shall exclude from our considerations "constricted" passages of the type shown in Fig. 2, because for these

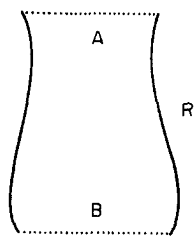


FIG. 2. Radiation passage for which neither E_{A-A} nor E_{B-B} vanish.

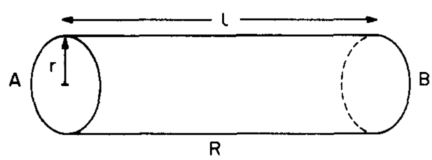


FIG. 3. Cylindrical radiation passage of length l and radius r.

neither E_{B-B} nor E_{A-A} vanish and our technique does not yield any useful information. Then the relation analogous to equation (5) turns out to be

$$1 - E_{B-A}(\rho) = \varepsilon \frac{A_R}{A_B} E_{R-B}(\rho). \tag{7}$$

From the radiation balance between A and B, one further finds the wellknown reciprocity relation [3, 4]

$$A_A E_{A-B}(\rho) = A_B E_{B-A}(\rho). \tag{8}$$

In many cases, the quantities $E_{R-A}(\rho)$ and $E_{R-B}(\rho)$ turn out to be easy to calculate at the end points $\rho = 0$ and $\rho = 1$. At $\rho = 0$, they reduce to ordinary shape factors

$$E_{R-A}(0) = F_{R-A} \quad \text{and} \quad E_{R-B}(0) = F_{R-B}. \tag{9}$$

The calculation at $\rho = 1$ will be discussed in the following sections; for the moment we only point out the relation

$$E_{R-A}(1)+E_{R-B}(1) = 1 \tag{10}$$

which obviously follows from energy conservation if no radiation is trapped inside the cavity. As ρ increases, less of the radiation emitted by R is adsorbed on the way to A or B; therefore, $E_{R-A}(\rho)$ and $E_{R-B}(\rho)$ grow monotonically with ρ. By virtue of equations (5)–(8), their values at $\rho = 0$ and $\rho = 1$ provide rigorous upper and lower bounds for all five exchange factors.

Furthermore, for small ε, the exchange factor $E_{R-B}(\rho)$ becomes equal to $E_{R-B}(1)-0(\varepsilon)$. Therefore, equation (7) becomes, in the same limit,

$$1-E_{B-A}(\rho) = \varepsilon\frac{A_R}{A_B}E_{R-B}(1)-O(\varepsilon^2);$$

in other words, knowledge of $E_{R-B}(1)$ determines $E_{B-A}(\rho)$ to order ε. In a similar manner, equations (5), (8) and (10) determine $E_{A\to B}(\rho)$ and $E_{A\to A}(\rho)$ order ε.

The upper bounds on the exchange factors can be improved even further (by what amounts to another order in ε, for most configurations). The exchange factor $E_{R-B}(\rho)$ can be expanded in a power series

$$E_{R-B}(\rho) = f_0+\rho f_1+\rho^2 f_2+\ldots \tag{11}$$

where $\rho^n f_n$ is the fraction of the diffuse radiation from R which reaches B via n specular reflections. Since $\rho^n \leqslant \rho$, this implies the inequality

$$E_{R-B}(\rho) \leqslant f_0 + \rho(f_1+f_2+f_3+\ldots)$$

which can be rewritten as (note $f_0 = F_{R-B}$)

$$E_{R-B}(\rho) \leqslant E_{R-B}(1)-\varepsilon[E_{R-B}(1)-F_{R-B}]. \tag{12}$$

From this follows by use of equation (7) the rigorous lower bound $\tau_{\text{l.b.}}$ for E_{B-A}

$$E_{B-A}(\rho) \geqslant \tau_{\text{l.b.}} = 1-\varepsilon\frac{A_R}{A_B}E_{R-B}(1) + \varepsilon^2\frac{A_R}{A_B}[E_{R-B}(1)-F_{R-B}]. \tag{13}$$

In order to provide both an intuitive interpretation and a simpler one-parameter approximation for the exchange factor $E_{B-A}(\rho)$, we consider the average number of reflections $\langle n\rangle_{B-A}$ which radiation has to undergo on its passage from B to A. Clearly, if all rays make n reflections, the exchange factor must be equal to ρ^n. For the general case, however, one has to write

$$E_{B-A}(\rho) = \sum_{n=0}^{\infty}\rho^n P_{B-A}(n), \tag{14}$$

where $P_{B-A}(n)$ is the probability that diffuse radiation emitted by B make exactly n reflections before reaching A. Let us convert this power series in ρ to one in $\varepsilon = 1-\rho$. Using the binomial expansion to rewrite equation (14) as

$$E_{B-A}(\rho) = \sum_{n=0}^{\infty}P_{B-A}(n)\sum_{k=0}^{n}\binom{n}{k}(-\varepsilon)^k \tag{15}$$

one obtains, after interchanging the order of summation,

$$E_{B-A}(\rho) = \sum_{k=0}^{\infty}\left\langle\binom{n}{k}\right\rangle_{B-A}(-\varepsilon)^k = 1-\varepsilon\langle n\rangle_{B-A}+O(\varepsilon^2) \tag{16}$$

where the expectation values are defined by

$$\left\langle\binom{n}{k}\right\rangle_{B-A} = \sum_{n=0}^{\infty}P_{B-A}(n)\binom{n}{k}. \tag{17}$$

Expanding in a similar fashion, the quantity

$$\rho^{\langle n\rangle} = 1-\langle n\rangle\varepsilon+\frac{\langle n\rangle^2}{2}\varepsilon^2-+\ldots \tag{18}$$

one learns that in general, the formula

$$E_{B-A}(\rho) \simeq \rho^{\langle n\rangle_{B-A}} \tag{19}$$

is valid only to lowest order in ε; it is exact if and only if $\langle n\rangle_{B-A}^k = \langle n^k\rangle_{B-A}$ for all $k = 0, 1, 2\ldots$. (Since $(\langle n^2\rangle - \langle n\rangle^2)^{1/2}$ is the standard deviation, this condition is satisfied only if all rays make the same number of reflections.) For many configurations $\langle n\rangle_{B-A}^2$ and $\langle n^2\rangle_{B-A}$ will not differ significantly and hence this simple formula can give good numerical results even for fairly large ε. We present some numerical examples in the next section.

The average number of reflections is found by combining equations (7) and (16) and taking the limit $\varepsilon \to 0$, with the result

$$\langle n\rangle_{B-A} = \frac{A_R}{A_B}E_{R-B}(1). \tag{20}$$

In some cases the interchange of summations* in equation (15) may not be justified. However, a slightly more complicated proof, presented in the appendix, shows that equation (20) holds whenever $P_{B-A}(n)$ decreases fast enough as $n\to\infty$ to make $\langle n\rangle_{B-A}$ finite.

In terms of $\langle n\rangle_{B-A}$, the lower bound (13) can be reexpressed in the form

$$E_{B-A}(\rho) \geqslant \tau_{\text{l.b.}} = 1-\varepsilon\langle n\rangle_{B-A} + \varepsilon^2[\langle n\rangle_{B-A}-F_{B-R}] \tag{21}$$

where F_{B-R} is an ordinary shape factor.

The exchange factor E_{A-A} can also be interpreted in terms of an average number of reflections. For this purpose, it is appropriate to define the average $\langle n\rangle_{A-A}$ with respect to that fraction of the diffuse radiation from A which can get back to A; this fraction is given by

$$E_{A-A}(1) = 1-\frac{A_B}{A_A}. \tag{22}$$

Equations (5), (7) and (8) can be combined to

$$E_{A-A}(\rho) = E_{A-A}(1)+\varepsilon\frac{A_R}{A_A}[E_{R-B}(\rho)-E_{R-A}(\rho)]. \tag{23}$$

*See any textbook on analysis, e.g. [5].

Comparison with

$$E_{A-A}(\rho) = E_{A-A}(1)\rho^{\langle n\rangle_{A-A}} \tag{24}$$

in the limit $\varepsilon \to 0$ yields the average number of reflections

$$\langle n\rangle_{A-A} = \frac{A_R}{A_A - A_B}\left[E_{R-B}(1) - E_{R-A}(1)\right]. \tag{25}$$

3. CYLINDRICAL RADIATION PASSAGE

It is instructive to compare the approximations developed in this paper with the exact solution for the case of the cylindrical passage, Fig. 3, one of the few nontrivial examples for which the exact solution is known [2]. Even though neither $E_{A-B}(\rho)$ nor $E_{R-B}(\rho)$ are listed explicitly in [2], they can easily be derived from the exchange factor $E_{\mathrm{d}R-A}$ for radiation from an infinitesimal ring element dR. The latter is given in equation (29) of [2]

$$E_{\mathrm{d}R-A} = \varepsilon \sum_{n=1}^{\infty} \rho^{n-1}\left\{\frac{\left(\dfrac{\eta_i}{2n}\right)^2 + \frac{1}{2}}{\left[\left(\dfrac{\eta_i}{2n}\right)^2 + 1\right]^{1/2}} - \frac{\eta_i}{2n}\right\} \tag{26}$$

where $\eta_i = l_i/r$ is the coordinate (along the cylinder axis) of dR. E_{R-A} is found by averaging over the entire wall R

$$E_{R-B} = E_{R-A} = \frac{1}{\eta}\int_0^{\eta} \mathrm{d}\eta_i E_{\mathrm{d}R-A}(\eta_i), \tag{27}$$

where $\eta = l/r$. Inserting equation (26) and integrating, one finds

$$E_{R-A}(\rho) = \tfrac{1}{2}\varepsilon \sum_{n=1}^{\infty} \rho^{n-1}\left\{\left[1 + \left(\frac{\eta}{2n}\right)^2\right]^{1/2} - \left(\frac{\eta}{2n}\right)\right\}. \tag{28}$$

Combined with equation (7), this yields the desired exchange factor $E_{A-B}(\rho)$ (= *transmission factor* τ)

$$\tau = E_{A-B}(\rho) = 1 - \eta\varepsilon^2 \sum_{n=1}^{\infty} \rho^{n-1} \times\left\{\left[1 + \left(\frac{\eta}{2n}\right)^2\right]^{1/2} - \left(\frac{\eta}{2n}\right)\right\}. \tag{29}$$

With regard to the approximations, we note that symmetry implies

$$E_{R-A}(\rho = 1) = \tfrac{1}{2}, \tag{30}$$

because in the limit $\rho \to 1$, all radiation emitted by R must escape either through A or through B. (Of course, the $\rho \to 1$ limit of equation (28) agrees with that.) Together with the area ratio

$$\frac{A_R}{A_A} = \frac{l}{r} = \eta \tag{31}$$

this implies, via equation (20), that the average number of reflections is

$$\langle n\rangle = \frac{l}{r} \tag{32}$$

for a specular cylindrical radiation passage of length l and radius r. Since the radiation shape factor F_{A-R} is

$$F_{A-R} = \eta(1+\eta^4/4)^{1/2} - \eta^2/2, \tag{33}$$

the lower bound (21) for the transmission factor is

$$E_{A-B}(\rho) \geqslant \tau_{\mathrm{l.b.}} = 1 - \eta\varepsilon + \varepsilon^2[\eta - \eta(1+\eta^2/4)^{1/2} + \eta^2/2]. \tag{34}$$

In Fig. 4, we have plotted the exact solution (29) as

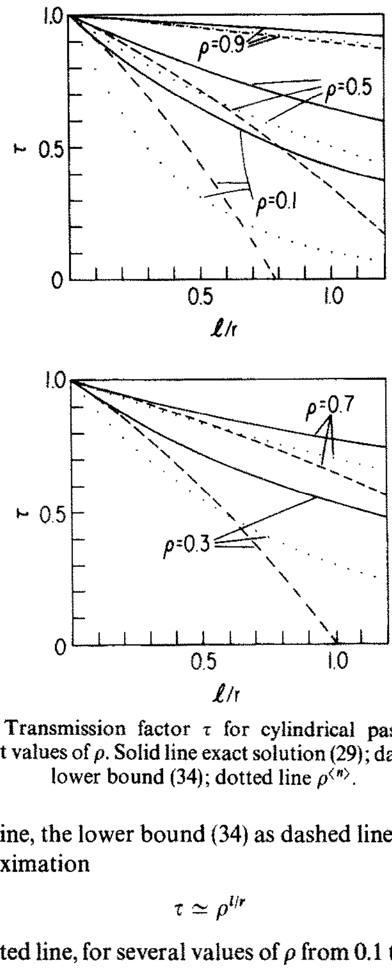


FIG. 4. Transmission factor τ for cylindrical passage for different values of ρ. Solid line exact solution (29); dashed line lower bound (34); dotted line $\rho^{\langle n\rangle}$.

solid line, the lower bound (34) as dashed line, and the approximation

$$\tau \simeq \rho^{l/r} \tag{35}$$

as dotted line, for several values of ρ from 0.1 to 0.9. As expected, the approximations are reliable when $\varepsilon\langle n\rangle$ is small. For example, if $\varepsilon\langle n\rangle \lesssim 0.1$, the simple formula $\rho^{\langle n\rangle}$ is accurate to at least 3% for $\varepsilon < 0.3$, while the lower bound (34) agrees with the exact answer to about 1% for all $\varepsilon > 0.3$.

4. COMPOUND PARABOLIC CONCENTRATOR

A radiation concentrator of considerable interest for solar energy collection is the compound parabolic concentrator [6–8]. It consists of two parabolic segments, arranged as shown in Fig. 5. Here we consider only the two-dimensional, or troughlike version. This concentrator has the property of accepting all radiation which hits the aperture A within the acceptance

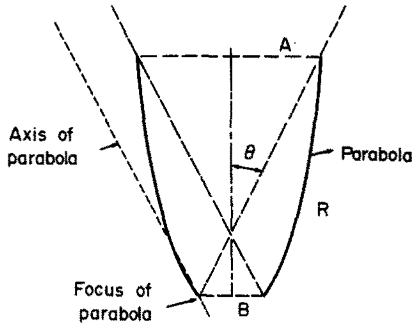


FIG. 5. Compound parabolic concentrator with acceptance half angle θ.

angle ($|\theta_{in}| \leqslant \theta$) and concentrating it by a factor

$$C = \frac{A_A}{A_B} = \frac{1}{\sin\theta}. \tag{36}$$

(The three-dimensional, or conelike version concentrates by $1/\sin^2\theta$.) (This is the maximum concentration permitted by the second law of thermodynamics [8–10].) All radiation outside the acceptance angle, i.e. $|\theta_{in}| > \theta$ is rejected as indicated in Fig. 6.

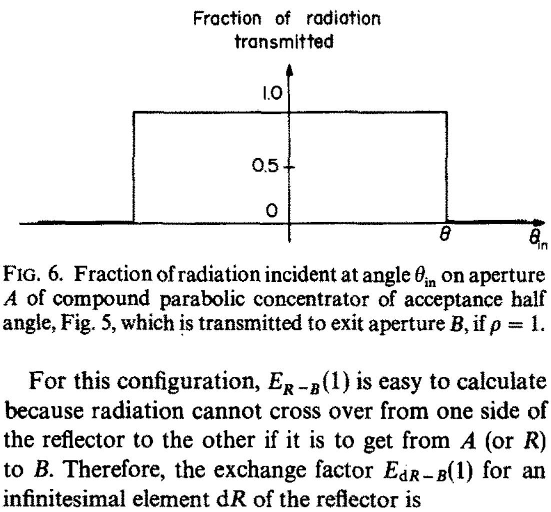


FIG. 6. Fraction of radiation incident at angle θ_{in} on aperture A of compound parabolic concentrator of acceptance half angle, Fig. 5, which is transmitted to exit aperture B, if $\rho = 1$.

For this configuration, $E_{R-B}(1)$ is easy to calculate because radiation cannot cross over from one side of the reflector to the other if it is to get from A (or R) to B. Therefore, the exchange factor $E_{dR-B}(1)$ for an infinitesimal element dR of the reflector is

$$E_{dR-B}(1) = 1/2\int_{\theta_c}^{\pi/2}\cos\theta'\,d\theta' = 1/2(1-\sin\theta_c), \tag{37}$$

θ_c being the angle between reflector normal at dR and the line from dR to the far edge of the absorber B as shown in Fig. 7. The complete exchange factor is found

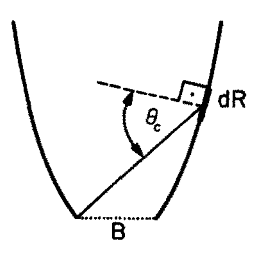


FIG. 7. Calculation of $E_{dR-B}(1)$ for compound parabolic concentrator.

by averaging $E_{dR-B}(1)$ over the whole reflector area.

$$E_{R-B}(1) = \frac{1}{A_R}\int_{A_R} dR\,E_{dR-B}(1). \tag{38}$$

For the integration, it is convenient to use a particular coordinate system, for example, cartesian coordinates with y axis along the parabola axis, but the result is, of course, coordinate independent:

$$E_{R-B}(1) = 1/2\left[1 - \frac{A_A}{A_R}\frac{(1-\sin\theta)(1+2\sin\theta)}{\sin\theta}\right]. \tag{39}$$

For the reflector area one finds,

$$A_R = A_A\sin\theta(1+\sin\theta)\left\{\frac{\cos\theta}{\sin^2\theta} + \log\left[\frac{(1+\sin\theta)(1+\cos\theta)}{\sin\theta\{\cos\theta+[2(1+\sin\theta)]^{1/2}\}}\right] - \frac{\sqrt{2}\cos\theta}{(1+\sin\theta)^{3/2}}\right\} \tag{40}$$

and hence the average number of reflections, as a function of the acceptance half angle θ, is given by

$$\langle n\rangle_{B-A} = 1/2(1+\sin\theta)\left\{\frac{\cos\theta}{\sin^2\theta} + \log\left[\frac{(1+\sin\theta)(1+\cos\theta)}{\sin\theta\{\cos\theta+[2(1+\sin\theta)]^{1/2}\}}\right] - \frac{\sqrt{2}\cos\theta}{(1+\sin\theta)^{3/2}}\right\} - \frac{(1-\sin\theta)(1+2\sin\theta)}{2\sin^2\theta}. \tag{41}$$

Incidentally, the $\theta \to 0$ limit of $\langle n\rangle$, for the configurations of both Fig. 5 and (iv) in Table 2, is

$$\langle n\rangle_{B-A} \to \frac{1}{2}\log\frac{1}{\theta}. \tag{42}$$

We have written a ray tracing computer program to calculate the exchange factor $E_{B-A}(\rho)$ for compound parabolic concentrators for various value of ρ. The results, displayed in Table 1, indicate that for moderate concentrations ($C \lesssim 10$, $\langle n\rangle \lesssim 1.3$) the simple formula

$$\tau = E_{B-A}(\rho) \simeq \rho^{\langle n\rangle_{B-A}}$$

is good to about 4% for ρ above 0.7; the lower bound (21) agrees with the exact answer to better than 7% for *all* values of ρ. The smaller $\langle n\rangle$, the better the approximation, for example, for $C \lesssim 6$, $\langle n\rangle \lesssim 1.0$, the lower bound (21) reproduces the exact answer within 2%, for any ρ.

Radiation incident on the aperture outside the acceptance angle ($|\theta_{in}| > \theta$) of a compound parabolic concentrator bounces back and forth between the reflector walls until it reemerges through the aperture. In order to estimate the fraction of this radiation which is absorbed by the reflector, we have calculated [11] the average number of reflections for these rays. It turns out to be

$$\langle n\rangle_{A-A} = 2 + 1/\sin\theta; \tag{43}$$

Table 1. Exchange factor $E_{B-A}(\rho)$ for compound parabolic concentrator as a function of reflectivity ρ and acceptance half angle θ (the left column also lists the concentration $C = 1/\sin\theta$ and the average number of reflections). For each θ and ρ the table lists three values for $E_{B-A}(\rho)$: the exact value (top), $\rho^{\langle n \rangle}$ (middle), and rigorous lower bound, equation (21) (bottom)

θ (C) $\langle n \rangle_{B-A}$	ρ 0.100	0.300	0.500	0.700	0.800	0.900	$0.950 = \rho$
5°	0.148	0.296	0.464	0.655	0.761	0.875	0.936
(11.47)	0.049	0.207	0.404	0.627	0.747	0.871	0.935
1.307	0.137	0.275	0.443	0.643	0.754	0.873	0.936
10°	0.221	0.373	0.535	0.709	0.802	0.899	0.949
(5.76)	0.091	0.285	0.485	0.689	0.792	0.896	0.948
1.044	0.216	0.364	0.526	0.704	0.799	0.898	0.948
20°	0.330	0.471	0.616	0.766	0.842	0.920	0.960
(2.92)	0.156	0.379	0.572	0.750	0.835	0.919	0.959
0.807	0.329	0.468	0.613	0.764	0.841	0.920	0.960
30°	0.417	0.543	0.671	0.801	0.867	0.933	0.966
(2.0)	0.212	0.444	0.627	0.786	0.860	0.931	0.966
0.674	0.416	0.542	0.670	0.800	0.866	0.933	0.966
40°	0.495	0.606	0.717	0.830	0.886	0.943	0.971
(1.56)	0.268	0.502	0.673	0.815	0.880	0.942	0.971
0.572	0.495	0.605	0.717	0.829	0.886	0.943	0.971
60°	0.655	0.732	0.808	0.885	0.923	0.962	0.981
(1.16)	0.412	0.629	0.766	0.872	0.918	0.960	0.980
0.385	0.655	0.732	0.808	0.885	0.923	0.962	0.981
80°	0.861	0.892	0.923	0.954	0.969	0.985	0.992
(1.02)	0.701	0.830	0.898	0.946	0.966	0.984	0.992
0.155	0.861	0.892	0.923	0.954	0.969	0.985	0.992

*For each θ and ρ: top entry = exact exchange factor; middle entry = $\rho^{\langle n \rangle}$; bottom entry = rigorous lower bound, equation (21).

of course the average has been taken over all diffuse radiation outside the acceptance angle, as implied in the discussion in Section 2.

In [11], further details can be found which are important in practice, in particular the effects of truncation (i.e. cutting off a portion of the reflector) and the transmission factor for collimated incident radiation.

5. OTHER EXAMPLES

In Table 2 the area ratios, the exchange factor $E_{R-B}(1)$ and the average number of reflections $\langle n \rangle_{A-B}$ are listed for a variety of specular passages, as sketched at the left end of each row:

(i) Parallel reflecting planes (of infinite extent in the direction perpendicular to the plane of the paper), length l and separation h.

(ii) Cylindrical segment, useful for piping diffuse radiation around corners.

(iii) V-trough, provided trough angle large enough so no radiation from B to A can cross over from one reflector side to the other.

(iv) Modified compound parabolic concentrator which concentrates radiation onto both sides of a fin; for details, see [8] or [12].

(v) Convolute reflector trough for circular tube. Entrance aperture stretches from left end of dotted line to right end of dotted line, and includes top portion of tube. Exit aperture B is surface of tube [13].

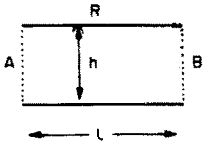
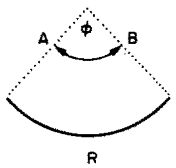
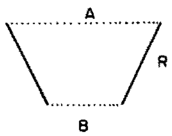
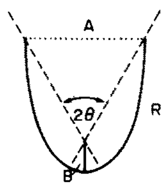
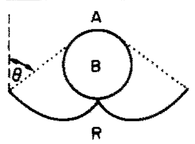
6. CONCLUSIONS

The examples described above indicate that the average number of reflections $\langle n \rangle$ is easy to calculate analytically for the following two types of specular radiation passages: (i) passages with sufficient symmetry such that $E_{R-A}(1) = E_{R-B}(1)$; and (ii) two-dimensional or troughlike passages where no radiation crosses over from one side of the reflector to the other on its way from A to B. Type (i) includes three-dimensional passages like Fig. 3, but with arbitrary rather than circular cross section. As further examples of type (ii) we mention reflector troughs consisting of straight segments, arranged such that no radiation cross over occurs. More general cases, for instance, conical passages of compound parabolic or of V-like profile, are more difficult to analyze, although sometimes good estimates for $E_{R-B}(1)$ can be obtained by means of shape factors.

Quite apart from the calculation of $\langle n \rangle$ itself, we have provided a framework for approximating the exchange (or transmission) factor $\tau = E_{B-A}(\rho)$ by means of simple one-parameter or two-parameter formulas, equations (19) and (21), respectively. These formulas will certainly provide a good approximation (to within a few percent) whenever τ is larger than about 0.9 or $\varepsilon\langle n \rangle$ smaller than about 0.1, and in some cases, good results are obtained even when $\varepsilon\langle n \rangle \gtrsim 1$.

This approach is particularly useful for the analysis of solar energy collectors because in this application

Table 2. Examples of two dimensional or troughlike radiation passages

Radiation passage	A_B/A_A	A_R/A_B	$E_{R-B}(1)$	$\langle n\rangle_{B-A}$
	1	$2l/h$	1/2	l/h
	1	ϕ	1/2	$\phi/2$
			F_{R-B}	F_{B-R}
	$1/\sin\theta$	$\dfrac{\cos\theta}{\sin^2\theta}+\log\left(\dfrac{1+\cos\theta}{\sin\theta}\right)$	$\frac{1}{2}-\dfrac{0.5\cot^2\theta}{\dfrac{\cos\theta}{\sin^2\theta}+\log\left(\dfrac{1+\cos\theta}{\sin\theta}\right)}$	$\frac{1}{2}\left[\dfrac{\cos\theta}{1+\cos\theta}+\log\left(\dfrac{1+\cos\theta}{\sin\theta}\right)\right]$
	1	$2\left(\theta+\frac{\pi}{2}\right)$	1/2	$\left(\theta+\frac{\pi}{2}\right)/4$

good optical efficiency demands that the transmission factor τ be reasonably close to one. Once $\langle n\rangle$ has been calculated (either for diffuse or for collimated radiation), τ can be taken as $\rho^{\langle n\rangle}$ without any need to repeat the calculation for each new value of ρ.

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APPENDIX

In order to derive equation (20) for the average number of reflections, we had assumed that the order of summations in equation (15) can be interchanged [5]. This is not justified if the series fails to converge absolutely. In fact, for configuration where $P_{B-A}(n)$ falls only like some power of n as $n \to \infty$ (for example like n^{-4} for the cylindrical segment of Section 5, (ii)) most of the expectation values $\langle\binom{n}{k}\rangle$ are infinite. In such cases, the power series in ε (16) for $E_{B-A}(\rho)$ is not well defined. We therefore present a proof which assumes only that the probabilities decrease fast enough at infinity to make $\langle n\rangle_{B-A}$ finite.

In the following, we omit the subscripts for simplicity. We start with equation (15), i.e. before changing the order of summations,

$$E(\rho) = \sum_{n=0}^{\infty} P(n) \sum_{k=0}^{n} \binom{n}{k}(-\varepsilon)^k. \tag{A1}$$

If $\langle n\rangle$ is finite, one can separate the $O(\varepsilon^0)$ and $O(\varepsilon^1)$ terms from the series and write

$$E(\rho) = \sum_{n=0}^{\infty} P(n)\left[1 - n\varepsilon + \sum_{k=2}^{n} \binom{n}{k}(-\varepsilon)^k\right]$$

$$= 1 - \varepsilon[\langle n\rangle + R(\varepsilon)] \tag{A2}$$

with a remainder

$$R(\varepsilon) \equiv \frac{1}{\varepsilon} \sum_{n=0}^{\infty} P(n) \sum_{k=2}^{n} \binom{n}{k}(-\varepsilon)^k. \tag{A3}$$

It is convenient to express $R(\varepsilon)$ in the form

$$R(\varepsilon) = \frac{1}{\varepsilon} \sum_{n=0}^{\infty} P(n)[(1-\varepsilon)^n - 1] + \sum_{n=0}^{\infty} nP(n). \tag{A4}$$

With the help of the identity

$$(1-\varepsilon)^n - 1 = \rho^n - 1 = -(1-\rho)(1+\rho+\rho^2+\ldots\rho^{n-1})$$

$$= -\varepsilon \sum_{k=0}^{n-1} \rho^k; \tag{A5}$$

this can be recast as

$$R(\varepsilon) = \sum_{n=0}^{\infty} P(n)\left[n - \sum_{k=0}^{n-1} \rho^k\right]. \tag{A6}$$

This series converges for $|\rho| \leq 1$ and at $\rho = 1$ it vanishes, hence

$$\lim_{\varepsilon \to 0} R(\varepsilon) = 0. \tag{A7}$$

Combined with equation (A2), the vanishing of $R(\varepsilon)$ at $\varepsilon \to 0$ implies the result

$$\lim_{\varepsilon \to 0} E_{B-A}(\rho) = 1 - \langle n\rangle_{B-A}\varepsilon. \tag{A8}$$

In the same limit equation (7) becomes

$$\lim_{\varepsilon \to 0} E_{B-A}(\rho) = 1 - \varepsilon \frac{A_R}{A_B} E_{R-B}(1), \tag{A9}$$

and comparison of the coefficients of ε yields equation (20) for $\langle n\rangle_{B-A}$.

TRANSFERT PAR RAYONNEMENT A TRAVERS LES PASSAGES SPECULAIRES—UNE APPROXIMATION SIMPLE

Résumé—On développe une nouvelle technique d'approximation des facteurs d'échange dans les passages réfléchissants. On démontre que pour un grand nombre de configurations, même celles présentant des surfaces réfléchissantes courbes, le nombre moyen de réflexions $\langle n\rangle$ peut être calculé à l'aide d'un formule analytique simple et sans aucun traçage de rayon. La fraction du rayonnement transmis à travers le passage, désignée par τ (un des facteurs d'échange), peut alors être approchée par $\tau \approx \rho^{\langle n\rangle}$ si le pouvoir réfléchissant ρ de la paroi du passage est élevé. On en déduit une limite inférieure rigoureuse qui est en accord avec le résultat exact à quelques pour cent près pour tout $\varepsilon = 1-\rho$, pourvu que $\varepsilon \langle n\rangle$ ne soit pas trop grand. Plusieurs exemples sont discutés, comprenant les passages cylindriques, sillons en V et concentrateurs paraboliques composés. La méthode est particulièrement utile pour le calcul de la transmission et l'absorption du rayonnement dans les concentrateurs solaires.

STRAHLUNGSAUSTAUSCH IN SPIEGELNDEN PASSAGEN

Zusammenfassung—Zur näherungsweisen Bestimmung der Austauschfaktoren für die Strahlung in spiegelnden Passagen wird eine neue Technik entwickelt. Es wird gezeigt, daß die mittlere Zahl von Reflektionen $\langle n\rangle$ für eine große Zahl von Konfigurationen selbst mit gekrümmten Reflektoroberflächen mit Hilfe einer einfachen analytischen Formel ohne Verfolgung des Strahlenganges berechnet werden kann. Der Anteil der durch die Passage durchgelassenen Strahlung τ (einer der Austauschfaktoren) kann durch $\tau \approx \rho^{\langle n\rangle}$ angenähert werden, wenn der Reflektionskoeffizient ρ der Passage groß ist. Es wird eine streng gültige untere Grenze abgeleitet, welche mit wenigen Prozent Abweichung für jedes $\varepsilon = 1-\rho$ mit dem exakten Ergebnis übereinstimmt, solange $\varepsilon\langle n\rangle$ nicht zu groß ist. Es werden mehrere Beispiele diskutiert, einschließlich zylindrischer und V-förmiger Passagen sowie zusammengesetzter Parabolspiegel. Die Methode eignet sich besonders zur Berechnung der Strahlungstransmission und -absorption in Sonnenkollektoren.

ЛУЧИСТЫЙ ПЕРЕНОС В ЗЕРКАЛЬНЫХ КАНАЛАХ. ПРОСТАЯ АППРОКСИМАЦИЯ

Аннотация — Разработан новый метод аппроксимации коэффициентов лучистого обмена зеркальных каналов. Показано, что для большинства конфигураций, даже при искривленной поверхности отражателя, среднее число отражений $\langle n\rangle$ можно рассчитать с помощью простой аналитической формулы, не прибегая к построению хода лучей. При больших значениях коэффициента зеркального отражения ρ стенки канала долю переданного по каналу излучения τ (один из коэффициентов обмена) можно аппроксимировать выражением $\tau \approx \rho^{\langle n\rangle}$. Определена точная нижняя граница, совпадающая в пределах нескольких процентов со строгим результатом для любой величины $\varepsilon = 1-\rho$ при условии не очень больших значений $\varepsilon\langle n\rangle$. Рассмотрено несколько примеров, включая цилиндрические и V-образные каналы и сложные параболические концентраторы. Метод в особенности удобен для расчета пропускания и поглощения излучения в солнечных аккумуляторах.